Fixed Points of Torus Action and Cohomology Ring of Toric Varieties

Kiumars Kaveh
Department of Mathematics
University of British Columbia

February 1, 2008

Abstract. Let X be a smooth simplicial toric variety. Let Z be the set of T-fixed points of X. We construct a filtration $F_0 \subset F_1 \subset \cdots$ of $A(Z)$, the ring of $\mathbb{C}$-valued functions on Z, such that $GrA(Z) \cong H^*(X, \mathbb{C})$ as graded algebras. This is the explanation of the general results of Carrell and Liebermann on the cohomology of T-varities, in the case of toric varieties. We give an explicit isomorphism between $GrA(Z)$ and Brion's description of the polytope algebra.

Key words: Toric variety, simple polytope, torus action, cohomology, polytope algebra.
Subject Classification: Primary 14M25; Secondary 52B20.

Contents

1 Introduction

In [3] and [4], Carrell and Lieberman prove that if X is a smooth projective variety over $\mathbb{C}$ with a holomorphic vector field $\mathcal{V}$ such that the $Zero(\mathcal{V})$ is non-trivial and isolated, then the coordinate ring $A(Z)$ of the zero scheme Z of $\mathcal{V}$ admits a filtration $F_0 \subset F_1 \subset \cdots$ such that the associated graded $GrA(Z)$ is isomorphic to $H^*(X, \mathbb{C})$ as graded algebra. In this paper, we give an explicit construction of this filtration in the toric case. We give an explicit isomorphism between $GrA(Z)$ and Brion's description of the polytope algebra (see [1]). We also give direct proofs that the usual relations in the cohomolgy of a toric variety hold in $GrA(Z)$.

In the toric case, for the vector field $\mathcal{V}$ one takes the generating vector field of a 1-parameter subgroup γ in general position of the torus T, so that the fixed point set of γ is the same as the fixed point set of T.

This paper is motivated in part by a comment of T. Oda. In [7] p.417, Oda comments about how to explain the results of Carrell-Lieberman in the toric case: as Khovanskii has shown in [6], composition of γ and the moment map of the toric variety X defines a Morse function on X whose critical points are the fixed points (see Remark 4.1). Since the number of critical points of index i is the i-th Betti number, Oda reasonably suggests that the grading on the fixed point set induced by the Morse index is the grading in Carrell-Lieberman and hence gives the cohomology algebra. It happens that this is not necessarily correct. One can see this in the example of $\mathbb{CP}^2$ (see Remark 6.1).

Acknowledgement: I would like to express my gratitude to Prof. James Carrell for suggesting the problem to me and helpful discussions. I would also like to thank my friend, Vladlen Timorin for helpful discussions.

2 Preliminaries on Cohomology of Varieties with G_m Action

In this section we briefly review the general theorems due to Carrell and Lieberman (see [2] and [4]) on the cohomology of varieties with a G_m action.

Let X be a smooth projective variety over $\mathbb{C}$.

Theorem 2.1 ([2], Theorem 5.4). *Suppose X admits a holomorphic vector field $\mathcal{V}$ with $Zero(\mathcal{V})$ isolated but non-trivial. Then the coordinate ring $A(Z)$*

of the zero scheme Z of $\mathcal{V}$ admits an increasing filtration $F_\bullet = F_\bullet A(Z)$ such that

(i) $F_i F_j \subset F_{i+j}$; *and*

(ii) $H^*(X,\mathbb{C}) = \bigoplus_{i\geq 0} H^{2i}(X,\mathbb{C}) \cong \bigoplus_{i\geq 0} Gr_{2i}(A(Z))$,

where the displayed summands are isomorphic over $\mathbb{C}$. Here

$$Gr_{2i}(A(Z)) := F_i A(Z)/F_{i-1}A(Z).$$

Let $E \to X$ be a holomorphic vector bundle and $\mathcal{E}$ its sheaf of holomorphic sections. One says that E is $\mathcal{V}$-equivariant if the derivation $\mathcal{V}$ of $\mathcal{O}_x$ lifts to $\mathcal{E}$. That is, there exists a $\mathbb{C}$-linear sheaf homomorphism $\tilde{\mathcal{V}} : \mathcal{E} \to \mathcal{E}$ such that if $\sigma \in \mathcal{E}_x$ and $f \in \mathcal{O}_{X,x}$ then

$$\tilde{\mathcal{V}}(f\sigma) = \mathcal{V}(f)\sigma + f\tilde{\mathcal{V}}(\sigma).$$

We then have:

Theorem 2.2 ([2], Theorem 5.5). *If p is a polynomial of degree l, then $p(\tilde{\mathcal{V}}_{|Z}) \in F_l A(Z)$, and in the associated graded, that is, in $Gr_{2l}A(Z)$, $p(\tilde{\mathcal{V}}_{|Z})$ corresponds to $p(c(\mathcal{E})) \in H^l(X,\Omega^l) = H^{2l}(X,\mathbb{C})$, where $c(\mathcal{E})$ denotes the Atiyah-Chern class of $\mathcal{E}$.*

3 Preliminaries on the Cohomology of Toric Varieties

Let T be the algebraic torus $(\mathbb{C}^*)^d$. As usual, N denotes the lattice of 1-parameter subgroups of T, $N_\mathbb{R}$ the real vector space $N \otimes_\mathbb{Z} \mathbb{R}$, M the dual lattice of N which is the lattice of characters of T and, $M_\mathbb{R}$ the real vector space $M \otimes_\mathbb{Z} \mathbb{R}$. A vector $n = (n_1,\ldots,n_d) \in \mathbb{Z}^d \cong N$ corresponds to the 1-parameter subgroup $t^n = (t^{n_1},\ldots,t^{n_d})$. Similarly, a covector $m = (m_1,\ldots,m_d) \in (\mathbb{Z}^d)^* \cong M$ corresponds to the character $x^m = x_1^{m_1}\cdots x_d^{m_d}$. We use $\langle\ ,\ \rangle : N \times M \to \mathbb{Z}$ for the natural pairing between N and M.

Let X be a d-dimensional smooth projective simplicial toric variety. Let $\Sigma \subset N_\mathbb{R}$ be the simplicial fan corresponding to X. We denote by $\Sigma(i)$ the set of all i-dimensional cones in Σ. For each $\rho \in \Sigma(1)$, let ξ_ρ be the primitive vector along ρ, i.e. the smallest integral vector on ρ.

There is a 1-1 correspondence between the orbits of dimension i in X and the cones in $\Sigma(d-i)$. The fixed points of T correspond to the cones in $\Sigma(d)$. In a smooth toric variety all the orbit closures are smooth, the cohomology class dual to the closure of the orbit corresponding to $\rho \in \Sigma(1)$ is denoted by $D_\rho \in H^2(X, \mathbb{C})$. It is well-known that the cohomology algebra of a toric variety is generated by the classes D_ρ. More precisely, we have:

Theorem 3.1 (see [5], p.106). *Let X be a smooth projective toric variety. Then $H^*(X, \mathbb{C}) = \mathbb{Z}[D_\rho, \rho \in \Sigma(1)]/I$, where I is the ideal generated by all*

(i) $D_{\rho_1} \cdot \ldots \cdot D_{\rho_k}, \quad \forall \rho_1, \ldots, \rho_k$ *not in a cone of* Σ*; and*

(ii) $\sum_{\rho \in \Sigma(1)} \langle \xi_\rho, u \rangle D_\rho, \quad \forall u \in M.$

Now, let $\Delta \subset M_{\mathbb{R}}$ be a simple rational polytope normal to the fan Σ. The polytope Δ defines a diagonal representation $\pi : T \to GL(V)$ where $\dim_{\mathbb{C}}(V) =$ the number of lattice points in Δ. If the mutual differences of the lattice points in Δ generate M then we get an embedding of X in $\mathbb{P}(V)$ as the closure of the orbit of $(1 : \ldots : 1)$. In the rest of the paper, we assume that the above condition holds for Δ.

The set of faces of dimension i in Δ is denoted by $\Delta(i)$. There is a 1-1 correspondence between the the faces in $\Delta(i)$ and the cones in $\Sigma(d-i)$ which in turn correspond to the orbits of dimension i in X. Hence the fixed points of T on X correspond to the vertices of Δ.

The support function $l_\Delta : N_{\mathbb{R}} \to \mathbb{R}$ is defined by: $l_\Delta(\xi) = \max_{x \in \Delta} \langle \xi, x \rangle$.

Let L_Δ be the line bundle on X obtained by restricting the universal subbundle on $\mathbb{P}(V)$ to X. We will need the following classical theorem which tells us how the first Chern class $c_1(L_\Delta)$ is represented as a linear combination of the classes D_ρ.

Theorem 3.2. *With notation as above we have*

$$c_1(L_\Delta) = \sum_{\rho \in \Sigma(1)} l_\Delta(\xi_\rho) D_\rho.$$

4 Main Theorem

As before, let X be a smooth projective simplicial toric variety with fan Σ and a polytope Δ normal to the fan which gives rise to a representation $\pi : T \to GL(V)$ and a T-equivariant embedding of X in $\mathbb{P}(V)$, for a vector

space V over $\mathbb{C}$. Let $\gamma \in N$ be a 1-parameter subgroup of T. We can choose γ so that the set of fixed points of γ is the same as the set of fixed points of T. We denote the set of fixed points by Z.

In this section, we construct a filtration $F_0 \subset F_1 \subset \cdots$ for $A(Z)$ such that $H^*(X, \mathbb{C}) \cong GrA(Z)$.

Notation: In the following, z denotes a fixed point, σ the corresponding d-dimensional cone in Σ and v the corresponding vertex in Δ. A 1-dimensional cone in Σ is denoted by ρ and the corresponding facet of Δ by F.

From Theorem 2.1 applied to the generating vector field of γ, there exists a filtration $F_0 \subset F_1 \subset \cdots$ of $A(Z)$, the ring of $\mathbb{C}$ valued functions on Z, so that $H^*(X, \mathbb{C}) \cong \bigoplus_{i=0}^{\infty} F_{i+1}/F_i$, as graded algebras. In particular, we have $H^2(X, \mathbb{C}) \cong F_1/F_0$. The subspace $F_0A(Z)$ is just the set of constant functions. To determine the image of $H^2(X, \mathbb{C})$ in $GrA(Z)$ we need to determine F_1. We start by finding the representatives in F_1 for the Chern classes of the line bundles.

The 1-parameter subgroup $\gamma : \mathbb{C}^* \to T$ acts on V via π and hence the action of γ on X lifts to an action of γ on the line bundle L_Δ. Thus the generating vector field of γ has a lift to L_Δ. If we view L_Δ as $\{(x, l) \in X \times V \mid x = [l]\}$ then the action of γ on L_Δ is given by:

$$\gamma(t) \cdot (x, l) = (\pi(t^\gamma)x, \pi(t^\gamma)l).$$

Now, from Theorem 2.2 we have:

Proposition 4.1. *Under the isomorphism $F_1/F_0 \cong H^2(X, \mathbb{C})$, the first Chern class $c_1(L_\Delta)$ is represented by the function f_Δ defined by:*

$$f_\Delta(z) = \langle \gamma, v \rangle, \quad \forall z \in Z,$$

where v is the vertex of Δ corresponding to the fixed point z.

Proof. In Theorem 2.2, take E to be L_Δ and p be the identity polynomial. The derivation $\tilde{\mathcal{V}}$ is just the derivation given by the G_m-action of γ on L_Δ. Let z be a fixed point and $(z, l) \in (L_\Delta)_z$ a point in the fiber of z. We have:

$$\begin{aligned} \gamma(t) \cdot (z, v) &= (z, \pi(t^\gamma)l), \\ &= (z, \langle \gamma, v \rangle l). \end{aligned}$$

and hence $f_\Delta(z) = \langle \gamma, v \rangle$. ∎

Next, we wish to determine the images of the classes $D_\rho, \rho \in \Sigma(1)$, in F_1/F_0. Fix a 1-dimensional cone ρ in $\Sigma(1)$. Let F be the facet of Δ orthogonal to ρ. We move the facet F of Δ parallely to obtain a new polytope Δ' (Figure 1). The polytope Δ' is still normal to the fan Σ. Let F' denote the facet of Δ' obtained by moving F. The maximum of the function $\langle \xi_\rho, \cdot \rangle$ on Δ and Δ' is obtained on the facets F and F' respectively. For support functions of these polytopes we can write:

$$l_\Delta(\xi_\rho) = \langle \xi_\rho, \text{ some point in } F\rangle,$$

$$l_{\Delta'}(\xi_\rho) = \langle \xi_\rho, \text{ some point in } F'\rangle$$

$$l_\Delta(\xi_{\rho'}) = l_{\Delta'}(\xi_{\rho'}), \quad \forall \rho' \neq \rho.$$

We also have:

$$c_1(L_\Delta) = l_\Delta(\xi_\rho)D_\rho + \sum_{\rho' \in \Sigma(1), \rho' \neq \rho} l_\Delta(\xi_{\rho'})D_{\rho'},$$

$$c_1(L_{\Delta'}) = l_{\Delta'}(\xi_\rho)D_\rho + \sum_{\rho' \in \Sigma(1), \rho' \neq \rho} l_{\Delta'}(\xi_{\rho'})D_{\rho'}.$$

Hence

$$c_1(L_\Delta) - c_1(L_{\Delta'}) = (l_\Delta(\xi_\rho) - l_{\Delta'}(\xi_\rho))D_\rho.$$

So

$$D_\rho = \frac{c_1(L_\Delta) - c_1(L_{\Delta'})}{l_\Delta(\xi_\rho) - l_{\Delta'}(\xi_\rho)}.$$

Now, let z be a fixed point, σ the corresponding d-dimensional cone, and v and v' the corresponding vertices in Δ and Δ' respectively. From Proposition 4.1, D_ρ corresponds to the function $f_\rho \in F_1A(Z)$ given by:

$$\begin{aligned} f_\rho(z) &= \frac{f_\Delta(z) - f_{\Delta'}(z)}{l_\Delta(\xi_\rho) - l_{\Delta'}(\xi_\rho)}, \\ &= \frac{\langle \gamma, v - v'\rangle}{l_\Delta(\xi_\rho) - l_{\Delta'}(\xi_\rho)}. \end{aligned}$$

If $v \notin F$ then $v = v'$ and hence $f_\rho(z) = 0$. If $v \in F$ then $l_\Delta(\xi_\rho) = \langle \xi_\rho, v\rangle$ and $l_{\Delta'}(\xi_\rho) = \langle \xi_\rho, v'\rangle$. We obtain that:

$$f_\rho(z) = \begin{cases} \frac{\langle \gamma, v-v'\rangle}{\langle \xi_\rho, v-v'\rangle} & \text{if } v \in F \\ 0 \quad \text{if } v \notin F \end{cases}$$

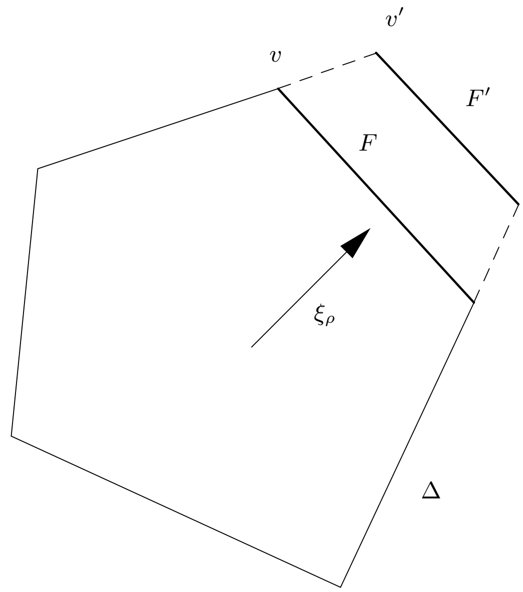

Figure 1: Moving facet F

Since Δ is a simple polytope, there are d edges at the vertex v. If $v \in F$, then there is only one edge e at v which does not belong to F. The vector $v - v'$, in fact, is along this edge. Note that the above formula for $f_\rho(z)$ does not depend on the length of the vector $v - v'$ (i.e. how much we move the facet F to obtain the new polytope Δ'). Let $u_{\sigma,\rho}$ be the vector along the edge e normalized such that $\langle u_{\sigma,\rho}, \xi_\rho \rangle = 1$. Then we have:

Proposition 4.2. *With notation as above, the cohomology class D_ρ is represented by the function f_ρ in $F_1A(Z)$ defined by*

$$f_\rho(z) = \begin{cases} \langle \gamma, u_{\sigma,\rho} \rangle & \text{if } v \in F \\ 0 & \text{if } v \notin F \end{cases}$$

Since $H^2(X, \mathbb{C})$ is generated by the classes $D_\rho, \rho \in \Sigma(1)$ and $H^*(X, \mathbb{C})$ is generated in degree 2, from Theorem 2.2 we obtain:

Theorem 4.3. $F_1A(Z)/F_0A(Z) = \mathrm{Span}_{\mathbb{C}}\langle f_\rho, \rho \in \Sigma(1) \rangle$. *Moreover, $F_iA(Z) =$ all polynomials of degree $\leq i$ in the f_ρ.*

One can prove directly that the functions $f_\rho, \rho \in \Sigma(1)$, satisfy the relations in the statement of Theorem 3.1. More precisely:

Theorem 4.4. *The functions $f_\rho, \rho \in \Sigma(1)$, satisfy the following relations:*

(i) $f_{\rho_1} \cdot \ldots \cdot f_{\rho_k} = 0, \quad \forall \rho_1, \ldots, \rho_k$ *not in a cone of* Σ*; and*

(ii) $\sum_{\rho\in\Sigma(1)} \langle \xi_\rho, u\rangle f_\rho =$ *some constant function on* $Z, \quad \forall u \in M.$

Proof. (i) is easy because every f_ρ is non-zero only at z such that the corresponding vertex lies in the facet F_ρ corresponding to ρ. Now, if $\rho_1, \ldots, \rho_k$ are not in a cone of Σ, it means that the the intersection of the corresponding facets F_{ρ_i} is empty, i.e. the product of the f_{ρ_i} is zero.

For (ii), let z be a fixed point and, σ and v the corresponding d-dimensional cone and vertex respectively. Let A be the $d \times d$ matrix whose rows are vectors ξ_ρ and let B be the $d \times d$ matrix whose columns are vectors $u_{\sigma,\rho}$, where ρ is an edge of σ. Since the cone at the vertex v, which is generated by the vectors $u_{\sigma,\rho}$, is dual to the cone σ, we get $AB = \mathrm{id}$. Now, we have

$$\begin{aligned} \sum_{\rho\in\Sigma(1)} \langle \xi_\rho, u\rangle f_\rho &= \sum_{\rho \text{ an edge of } \sigma} \langle \xi_\rho, u\rangle f_\rho \\ &= \sum_{\rho \text{ an edge of } \sigma} \langle \xi_\rho, u\rangle \langle \gamma, u_{v,F_\rho}\rangle \\ &= A \cdot u \cdot \gamma \cdot B, \end{aligned}$$

where $\cdot$ means product of matrices and γ is regarded as a row vector and u is regarded as a column vector. But the result of the above is simply $\langle \gamma, u\rangle$, since $AB = \mathrm{id}$. So we proved that the expression (ii) is independent of z and hence is a constant function on Z. □

One can introduce a finite affine set $\mathcal{Z}$ isomorphic to Z such that the natural grading on the coordinate ring $A(\mathcal{Z})$ coincides with the above filtration $F_\bullet$ given by the f_ρ. Define the function $\Theta : Z \to \mathbb{R}^{\Sigma(1)} \subset \mathbb{C}^{\Sigma(1)}$ by

$$\Theta(z)_\rho = f_\rho(z),$$

and let $\mathcal{Z} = \Theta(Z)$.

Proposition 4.5. *$Gr A(\mathcal{Z}) \cong H^*(X, \mathbb{C})$, as graded algebras. The grading on $A(\mathcal{Z})$ is induced from the usual grading of the polynomial algebra.*

Proof. Immediate. □

Remark 4.1. Let $\mu : X \to M_{\mathbb{R}}$ be the moment map of the toric variety and, as before, $\gamma \in N$ a 1-parameter subgroup in general position. In [6] Khovanskii shows that the composition of γ and μ defines a Morse function on X whose critical points are the fixed points of X. The Morse index of a fixed point corresponding to a vertex v is twice the number of edges at v on which the linear function γ is decreasing. Back to the definition of the functions f_ρ (Proposition 4.2), the linear function γ is decreasing on the edge e at v if and only if $f_\rho(z) < 0$. That is, the Morse index of a fixed point z is equal to twice the number of negative coordinates of the point $\Theta(z) \in \mathbb{R}^{\Sigma(1)}$. Since the number of critical points of index $2i$ is the $2i$-th Betti number of X, we conclude the non-trivial relation that: the number of points in $\mathcal{Z}$ exactly i of their coordinates are negative is equal to $\dim Gr_i A(\mathcal{Z})$.

5 Relation with the Polytope Algebra

To each simplicial polytope Δ, one can associate an algebra, called the *polytope algebra* of Δ (see [8], and for a more detailed explanation [9]). The direct limit of these algebras for all Δ is the McMullen's polytope algebra. McMullen's polytope algebra plays an important role in the study of finitely additive measures on the convex polytopes. For an integral polytope Δ, its polytope algebra coincides with the cohomology algebra of the corresponding toric variety X.

In [1], Brion gives a description of the polytope algebra of a polytope as a quotient of the algebra of continuous piecewise polynomial functions: let $\Sigma \subset N_{\mathbb{R}}$ be the fan of the polytope Δ. Let R be the algebra of all continuous functions on $N_{\mathbb{R}}$ which restricted to each cone of Σ are given by a polynomial. Let I be the ideal of R generated by all the linear functions on $N_{\mathbb{R}}$, then the polytope algebra of Δ is isomorphic to R/I.

There is a good set of generators for R parameterized by the set of 1-dimensional cones $\Sigma(1)$. For each $\rho \in \Sigma(1)$, define $g_\rho : N_{\mathbb{R}} \to \mathbb{R}$ as a piecewise linear function, supported on the cones containing ρ, as follows:

(i) $g_\rho = 0$ on any cone not containing ρ; and

(ii) for a d-dimensional cone σ containing ρ, the function g_ρ restricted to σ is the unique linear function defined by $g_\rho(x) = 0$ for $x \in \rho' \neq \rho, \rho' \in \Sigma(1)$ and $g_\rho(\xi_\rho) = 1$.

One can show that the g_ρ are a set of generators for R. Moreover, by sending g_ρ to D_ρ, we get an isomorphism between R/I and $H^*(X, \mathbb{C})$, in particular the g_ρ satisfy the relations in Theorem 3.1.

In the next theorem, we show how this description of the cohomology is related to the $GrA(Z)$ description:

Let γ be a 1-parameter subgroup in general position. Let $p \in R$ be a continuous piecewise polynomial of degree n. Define

$$\Phi(p) = \frac{\partial^n p}{\partial \gamma^n},$$

where $\partial^n/\partial\gamma^n$ means n times differentiation in the direction of the vector γ. Then $\Phi(p)$ is a constant function on each d-dimensional cone and hence can be viewed as a function in $A(Z)$. We have:

Theorem 5.1. *(i) $\Phi(g_\rho) = f_\rho$; and*

(ii) Φ induces an isomorphism between R/I and $GrA(Z)$.

Proof. (i) Let σ be a d-dimensional cone containing ρ. Since σ is simplicial the set $\{\xi_{\rho'} \mid \rho' \subset \sigma\}$ form a basis for $N_\mathbb{R}$. Consider the linear function l defined by $l(\xi_\rho) = 1$ and $l(\xi_{\rho'}) = 0, \rho' \subset \sigma$ and $\rho' \neq \rho$). Let A be the $d \times d$ matrix whose rows are vectors ξ_ρ and B be the $d \times d$ matrix whose columns are vectors $u_{\sigma,\rho}$, where ρ is an edge of σ. Let v be the vertex of Δ corresponding to σ. The cone at v is dual to σ and hence we have $AB = \mathrm{id}$. View γ as a row vector. The γ in the basis $\xi_{\rho'}, \rho' \subset \sigma$ is $\gamma A^{-1} = \gamma B$. Thus, one sees that the derivative of l along γ is equal to the ρ-th component of γB. But this is the same as $f_\rho(z)$.

For (ii), note that Φ is clearly an additive homomorphism. Let p and q $\in R$ be of degrees n and m respectively. Since $n+1$-th derivative of p and $m+1$-th derivative of q along γ are zero, one can see that $\Phi(pq) = \Phi(p)\Phi(q)$. Let p be a linear function on $N_\mathbb{R}$, then the first derivative of p along γ is a constant function hence $\Phi(p)$ is zero in $GrA(Z)$, i.e. Φ is well-defined on R/I. Since the n-th graded piece of R (respectively $A(Z)$) is the set of polynomials of degree n in the g_ρ (respectively f_ρ), from (i) we see that Φ : $R/I \to GrA(Z)$, is onto. Since $\dim(R/I)_i = \dim H^{2i}(X, \mathbb{C}) = \dim F_{i+1}/F_i$, it follows that Φ is 1-1 as well. This finishes the proof. $\square$

6 Examles

In this section we consider two examples in dimension 2, namely, $\mathbb{C}P^2$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$. For each example, we compute the functions f_ρ and the finite affine set $\mathcal{Z}$.

Example 6.1 ($X = \mathbb{C}P^2$). Fan Σ of $\mathbb{C}P^2$ is shown in Figure 2. There are 3 one dimensional cones denoted by ρ_1, ρ_2 and ρ_3. The primitive vectors along the ρ_i are $\xi_1 = (1,0), \xi_2 = (0,1)$ and $\xi_3 = (-1,-1)$. The vertices of a normal polytope to the fan are $v_1 = (1,1), v_2 = (-2,1)$ and $v_3 = (1,-2)$ (see Figure 3). They correspond to the three fixed points z_1, z_2 and z_3. At each vertex, there are two vectors along the edges. For v_1, we take $\{(1,0),(0,1)\}$, for v_2 we take $\{(-1,0),(-1,1)\}$ and finally, for v_3 we take $\{(0,-1),(1,-1)\}$.

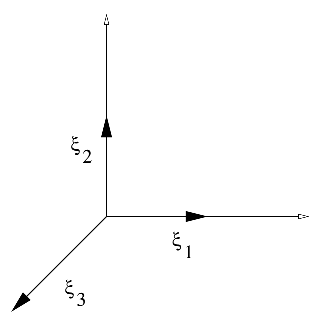

Figure 2: Fan of $\mathbb{C}P^2$

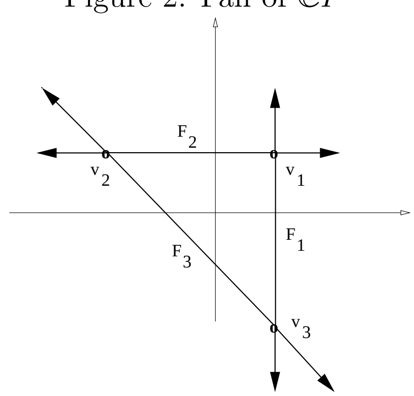

Figure 3: Polytope normal to the fan of $\mathbb{C}P^2$ and the vectors $u_{\sigma,\rho}$.

Let $\gamma = (\gamma_1, \gamma_2)$ be a 1-parameter subgroup. From the definition of the

functions f_ρ (Proposition 4.2), we get the following table for their values:

	z_1	z_2	z_3
f_1	γ_2	$\gamma_2-\gamma_1$	0
f_2	γ_1	0	$\gamma_1-\gamma_2$
f_3	0	$-\gamma_1$	$-\gamma_2$

and hence, $\mathcal{Z} = \{(\gamma_2,\gamma_1,0),(\gamma_2-\gamma_1,0,-\gamma_1),(0,\gamma_1-\gamma_2,-\gamma_2)\} \subset \mathbb{R}^3$. Note that the points in $\mathcal{Z}$ lie on the same line parallel to $(1,1,1)$. One can see that $Gr_iA(\mathcal{Z}) \cong \mathbb{C}, 0 \leq i \leq 2$ and $Gr_iA(\mathcal{Z}) = \{0\}, i > 2$. If x is a non-zero element of $Gr_1A(\mathcal{Z})$ then, $H^*(\mathbb{C}P^2,\mathbb{C}) \cong GrA(\mathcal{Z}) \cong \mathbb{C}[x]/\langle x^3\rangle$.

The above calculation can be carried out in general for $\mathbb{C}P^n$. One can show that all the points in the set $\mathcal{Z}$ lie on the same line parallel to $(1,\ldots,1)$, and hence $Gr_i \cong \mathbb{C}$ for $0 \leq i \leq n$ and $Gr_i \cong 0$ for $i > n$ and thus $H^*(\mathbb{C}P^n,\mathbb{C}) \cong GrA(\mathcal{Z}) \cong \mathbb{C}[x]/\langle x^{n+1}\rangle$. In fact, any set of n points lying on the same line can give the cohomology of $\mathbb{C}P^n$.

Remark 6.1. Consider the polytope Δ for $\mathbb{C}P^2$. For a γ in general position, there is one vertex of index 4, one vertex of index 2 and one vertex of index 0. Without loss of generality, assume that the indices of v_1, v_2 and v_3 are $0, 2$ and 4 respectively. Now, if the grading on $A(Z)$ is induced by the Morse index, the subspace of elements of degree ≤ 1 is generated by the functions supported on v_2, the only fixed point of index 2, and the constant functions. Hence, any function of degree ≤ 1 should have the same value on v_1 and v_3. But, for example, f_1 does not have this property while it is of degree ≤ 1. This shows that the grading by the Morse index does not coincide with the filtration generated by the f_ρ.

Example 6.2 ($X = \mathbb{C}P^1 \times \mathbb{C}P^1$). Fan Σ of $\mathbb{C}P^1 \times \mathbb{C}P^1$ is shown in Figure 4. There are 4 one dimensional cones denoted by ρ_1, ρ_2, ρ_3 and ρ_4. The primitive vectors along the ρ_i are $\xi_1 = (1,0), \xi_2 = (0,1), \xi_3 = (-1,0)$ and $\xi_4 = (0,-1)$. The vertices of a normal polytope to the fan are $v_1 = (1,1), v_2 = (-1,1), v_3 = (-1,-1)$ and $v_4 = (1,-1)$ (see Figure 5). They correspond to the four fixed points z_1, z_2, z_3 and z_4. At each vertex, there are two vectors along the edges. For v_1, we take $\{(1,0),(0,1)\}$, for v_2 we take $\{(-1,0),(0,1)\}$, for v_3 we take $\{(-1,0),(0,-1)\}$ and finally, for v_4 we take $\{(1,0),(0,-1)\}$.

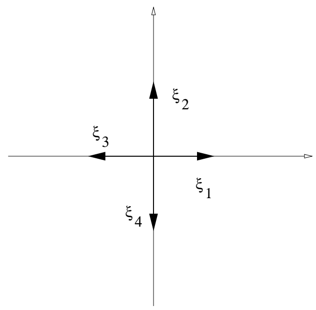

Figure 4: Fan of $\mathbb{C}P^1 \times \mathbb{C}P^1$

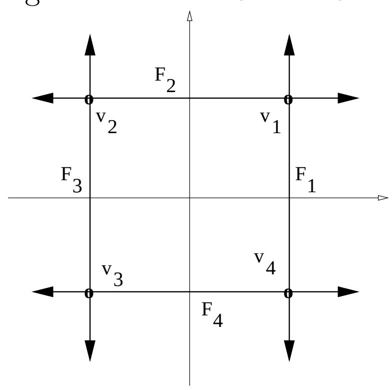

Figure 5: Polytope normal to the fan of $\mathbb{C}P^1 \times \mathbb{C}P^1$ and the vecotrs $u_{\sigma,\rho}$.

Let $\gamma = (\gamma_1, \gamma_2)$ be a 1-parameter subgroup. We get the following table for the values of the f_ρ:

	z_1	z_2	z_3	z_4
f_1	γ_1	0	0	γ_1
f_2	γ_2	γ_2	0	0
f_3	0	$-\gamma_1$	$-\gamma_1$	0
f_4	0	0	$-\gamma_2$	$-\gamma_2$

and hence, $\mathcal{Z} = \{(\gamma_1, \gamma_2, 0, 0), (0, \gamma_2, -\gamma_1, 0), (0, 0, -\gamma_1, -\gamma_2), (\gamma_1, 0, 0, -\gamma_2)\} \subset$

$\mathbb{R}^4$. Note that the points in $\mathcal{Z}$ lie on the same 2-plane defined by $f_1 - f_3 = \gamma_1$ and $f_2 - f_4 = \gamma_2$. Also, no three of them are colinear. Thus, one can see that $Gr_0 \cong \mathbb{C}, Gr_1 \cong \mathbb{C}^2, Gr_2 \cong \mathbb{C}$ and $Gr_i = \{0\}, i > 2$. If $\{x, y\}$ is a basis for $Gr_1 A(\mathcal{Z})$ then, $H^*(\mathbb{C}P^1 \times \mathbb{C}P^1, \mathbb{C}) \cong GrA(\mathcal{Z}) \cong \mathbb{C}[x, y]/\langle x^2, y^2 \rangle$. In fact, any set of 4 points lying on the same 2-plane such that no three are colinear can give the cohomology of $\mathbb{C}P^1 \times \mathbb{C}P^1$.

References

[1] Brion, M. *The structure of the polytope algebra.* Tohoku Math. J. (2) 49 (1997), no. 1, 1–32.

[2] Carrell, J.B. *Torus actions and cohomology* in Algebraic quotients. Torus actions and cohomology. The adjoint representation and the adjoint action, 83–158, Encyclopaedia Math. Sci., 131, Springer, Berlin, 2002.

[3] Carrell, J. B.; Lieberman, D. I. *Vector fields, Chern classes, and cohomology.* Several complex variables (Proc. Sympos. Pure Math., Vol. XXX, Part 1, Williams Coll., Williamstown, Mass., 1975), pp. 251–254. Amer. Math. Soc., Providence, R.I., 1977.

[4] Carrell, J. B.; Lieberman, D. I. *Vector fields and Chern numbers.* Math. Ann. 225 (1977), no. 3, 263–273.

[5] Fulton, W. *Introduction to toric varieties.* Annals of Mathematics Studies, 131. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.

[6] Khovanskii, A. G. *Hyperplane sections of polyhedra, toric varieties and discrete groups in Lobachevskiĭspace.* Funktsional. Anal. i Prilozhen. 20 (1986), no. 1, 50–61, 96.

[7] Oda, T. *Geometry of toric varieties.* Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), 407–440, Manoj Prakashan, Madras, 1991.

[8] Pukhlikov, A. V.; Khovanskiĭ, A. G. *The Riemann-Roch theorem for integrals and sums of quasipolynomials on virtual polytopes.* Algebra i Analiz 4 (1992), no. 4, 188–216; translation in St. Petersburg Math. J. 4 (1993), no. 4, 789–812

[9] Timorin, V. A. *An analogue of the Hodge-Riemann relations for simple convex polyhedra*. Uspekhi Mat. Nauk 54 (1999), no. 2(326), 113–162; translation in Russian Math. Surveys 54 (1999), no. 2, 381–426

University of British Columbia, Vancouver, B.C.
Email address: kaveh@math.ubc.ca